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α-Fair Power Allocation in Spectrum-Sharing Networks

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***Abstract*—To efficiently trade off system sum-rate and link fairness, this paper is dedicated to maximizing the sum of α-fair utility in spectrum-sharing networks, where multiple interfering links share one channel. Whereas three special cases including $\alpha = 0$ (sum-rate maximization), $\alpha = 1$ (proportional fairness) and $\alpha = \infty$ (max-min fairness) have been investigated in the literature, the complexity for cases $1 < \alpha < \infty$ and $0 < \alpha < 1$ is still unknown. In this paper, we prove that the problem is convex when $1 < \alpha < \infty$ and is NP-hard when $0 < \alpha < 1$. To deal with the latter case, we transform the objective function into a D.C. (difference of two concave functions) function. Then, a power allocation algorithm is proposed with fast convergence to a local optimal point. Simulation results show that the proposed algorithm can obtain global optimality in two-link cases when $0 < \alpha < 1$. In addition, we can get a flexible tradeoff between sum-rate and fairness in terms of Jain's index by adjusting α.**

***Index Terms*—Power allocation, sum-rate, α-fair, tradeoff, Jain's index**

I. Introduction

THE broadcast nature of wireless communications brings about serious co-channel interference and limits the system performance [1]. In such kind of spectrum-sharing systems, power allocation (PA) is a crucial issue for co-channel interference management, which can further be applied to improve system sum-rate, enhance link fairness and simultaneously prolong battery life of mobile terminals [2].

The sum-rate and the fairness of all links are two crucial performance indexes which need to be improved or maximized. Towards these two ends, the sum-rate and min-rate maximization problems have been well studied. Here, min-rate maximization, usually referred to as max-min fairness, can lead to the fairest rates of all links. For a network with multiple links and only one tone (frequency band), it has been shown in [3] by Luo that the sum-rate maximization problem is NP-hard, while max-min fairness can be achieved through linear programming (LP). Moreover, for the sum-rate maximization problem, many other works focused on designing algorithms either to find local optimality with low complexity or global optimality with high complexity [1], [2], [4]–[8].

The sum-rate and the degree of fairness are just like the two ends of a seesaw: improving one will often decrease the other [9]. In other words, maximizing sum-rate usually leads to the most unfair case, while min-rate maximization often results in low sum-rate. Therefore, an efficient tradeoff between sum-rate and fairness is also needed. In the literature, proportional fairness has been achieved by maximizing the sum-utility of all links, where each link's utility is the logarithm of its received rate. Also in [3], Luo has shown that this is a convex optimization problem when there are multiple links and only one tone.

Nevertheless, as a special tradeoff point, proportional fairness can not offer a smooth tradeoff between sum-rate and fairness. For this purpose, α-fair ($\alpha > 0$) power allocation may be a candidate, which can be achieved by maximizing the sum of α-fair utilities of all links [9]. As a larger α corresponds to a more fair case, varying α from zero to ∞ can smoothly control the tradeoff from the most unfair case to the most fair one. What should be emphasized is that $\alpha = 0$, $\alpha = 1$ and $\alpha = \infty$ just correspond to sum-rate maximization, proportional fairness and max-min fairness, respectively. As a result, in this paper, we focus on two intervals $1 < \alpha < \infty$ and $0 < \alpha < 1$ in spectrum-sharing networks, where multiple links utilize the same one frequency band. For power allocation with other utility functions and other network models, one can refer to [3], [5], [10]–[12] and references therein.

Although the objective function of the considered problem in this paper is not concave, it is still of great importance for us to figure out the complexity of the problem. This importance comes from the fact that a nonconvex problem at the first glance can be convex by appropriate reformulation [3], [8], [12]. Actually, we show that it is a convex problem when $1 < \alpha < \infty$ and NP-hard when $0 < \alpha < 1$. For the case $0 < \alpha < 1$, we first transform the objective into a D.C. function, and then propose an iterative algorithm with fast convergence to a local optimal point, where a convex optimization is carried out in each iteration.

The contributions of this paper are threefold. First, for $1 < \alpha < \infty$, we prove that the α-fair PA problem is convex. Second, for $0 < \alpha < 1$, we prove that the α-fair PA problem is NP-hard. Third, for the NP-hard case $0 < \alpha < 1$, we propose an efficient algorithm with fast convergence to a local optimal point. In particular, the algorithm can lead to the global optimality in two-link cases.

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The rest of this paper is organized as follows. In Section II, the system model is characterized and the considered problem is formulated. Section III discusses the complexity of the problem for cases $1 < \alpha < \infty$ and $0 < \alpha < 1$. Section IV presents an efficient algorithm for the case $0 < \alpha < 1$. In Section V, some simulation results are given. Finally, Section VI concludes this paper.

II. System Model and Problem Formulation

In this section, we first characterize the system model. Then, the α-fair PA problem is formulated.

A. System Model

As in many literatures, we consider a standard spectrum-sharing wireless network, where there are M distinct transmitter-receiver pairs (links) [3], [13]. All of the M links utilize the same system frequency band for achieving high spectrum efficiency. For simplicity, we normalize the system bandwidth as 1(Hz).

The channel fading coefficient from the transmitter of link i to the receiver of link j is denoted by $h_{i,j}$. All the channel fading coefficients are independent and identically distributed (i.i.d.) with $\mathcal{CN}(0,1)$. We consider a static snapshot of the network, i.e., $h_{i,j}$ remains constant during the observation time. Denote by $\boldsymbol{p} = (p_1, p_2, \cdots, p_M)^T$ the transmit power vector, in which p_i represents the transmit power of link i. Then, utilizing Shannon rate expression, we get the received rate of link i as

$$R_i(\boldsymbol{p}) = \log_2\left(1 + \frac{G_{i,i}p_i}{n_i + \sum_{j=1,j\neq i}^{M} G_{j,i}p_j}\right) \quad \text{bps}, \quad i = 1, 2, \cdots, M, \tag{1}$$

where $G_{j,i} = |h_{j,i}|^2$ and n_i is the receiving noise power at the receiver of link i.

B. Problem Formulation

We denote by $U_\alpha(x)(\alpha \geq 0)$ the α-fair utility function, which was firstly given in [14] as

$$U_\alpha(x) = \begin{cases} \ln(x), & \text{if } \alpha = 1 \\ \frac{1}{1-\alpha}x^{1-\alpha}, & \text{otherwise.} \end{cases} \tag{2}$$

Assume that link i receives utility $U_\alpha(R_i(\boldsymbol{p}))$ when its received rate is $R_i(\boldsymbol{p})$. Then, maximizing the sum-utility of all links results in α-fair PA. As has been pointed out in Section I, cases $\alpha = 0$, $\alpha = 1$ and $\alpha = \infty$ have been addressed. Therefore, we focus on the following α-fair PA problem for cases $1 < \alpha < \infty$ and $0 < \alpha < 1$:

$$\max_{\boldsymbol{p}} \quad \sum_{i=1}^{M} \frac{1}{1-\alpha}\left(\log_2\left(1 + \frac{G_{i,i}p_i}{n_i + \sum_{j=1,j\neq i}^{M} G_{j,i}p_j}\right)\right)^{1-\alpha} \tag{3}$$

$$\text{s.t.} \quad \text{C1}: 0 \leq p_i \leq p_i^{\max}, i = 1, 2, \cdots, M, \tag{3-a}$$

$$\text{C2}: \log_2\left(1 + \frac{G_{i,i}p_i}{n_i + \sum_{j=1,j\neq i}^{M} G_{j,i}p_j}\right) \geq R_i^{req}, \quad i = 1, 2, \cdots, M, \tag{3-b}$$

where $p_i^{\max}$ and R_i^{req} denote the maximum allowed transmit power and the minimum required data rate of link i, respectively. For simplicity, we denote by $\boldsymbol{p}^{\max} = (p_1^{\max}, p_2^{\max}, \cdots, p_M^{\max})^T$.

III. Complexity Discussions

In this section, we present the complexity analysis for cases $1 < \alpha < \infty$ and $0 < \alpha < 1$ individually.

A. Case $1 < \alpha < \infty$

The following theorem clarifies the complexity for this case.

Theorem 1. *The problem given in Eq.(3) is convex when $1 < \alpha < \infty$.*

Proof: We first replace the optimization variable p_i by $\exp(y_i)$. In addition, as α-fair utility function and $\exp(x)$ are monotonically increasing, we can introduce variables t_i $(i = 1, 2, \cdots, M)$ and transform the α-fair problem as

$$\max_{\boldsymbol{y},\boldsymbol{t}} \quad \sum_{i=1}^{M} \frac{1}{1-\alpha}(\exp(t_i))^{1-\alpha} \tag{4}$$

$$\text{s.t.} \quad \exp(y_i) \leq p_i^{\max}, i = 1, 2, \cdots, M, \tag{4-a}$$

$$\log_2\left(1 + \frac{G_{i,i}\exp(y_i)}{n_i + \sum_{j\neq i} G_{j,i}\exp(y_j)}\right) \geq R_i^{req}, \quad i = 1, 2, \cdots, M, \tag{4-b}$$

$$\log_2\left(1 + \frac{G_{i,i}\exp(y_i)}{n_i + \sum_{j\neq i} G_{j,i}\exp(y_j)}\right) \geq \exp(t_i), \quad i = 1, 2, \cdots, M, \tag{4-c}$$

where $\boldsymbol{y}$ and $\boldsymbol{t}$ are given by $\boldsymbol{y} = (y_1, y_2, \cdots, y_M)^T$ and $\boldsymbol{t} = (t_1, t_2, \cdots, t_M)^T$, respectively.

It can be easily checked that the objective function is concave when $1 < \alpha < \infty$ and the constraints in Eq.(4-a) and Eq.(4-b) are linear. As for Eq.(4-c), we can transform it into

$$\ln\left(\frac{n_i}{G_{i,i}}\exp(-y_i) + \sum_{j\neq i}\frac{G_{j,i}}{G_{i,i}}\exp(y_j - y_i)\right) + \ln\left(2^{\exp(t_i)} - 1\right) \leq 0, \quad i = 1, 2, \cdots, M. \tag{5}$$

Because the log-sum-exp function and $\ln\left(2^{\exp(t_i)} - 1\right)$ are convex [3], the constraint in Eq.(4-c) is also convex. As a result, the problem given in Eq.(3) is convex. ■

According to Theorem 1, the problem given in Eq.(3) can be efficiently solved when $1 < \alpha < \infty$ by standard solvers such as inter-point method [15].

B. Case $0 < \alpha < 1$

For this case, we present a theorem given as follows.

Theorem 2. *The problem given in Eq.(3) is NP-hard when $0 < \alpha < 1$.*

Proof: Considering $R_i^{req} = 0$, $p_i^{\max} = 1$ and $\frac{n_i}{G_{i,i}} = A$ for all links, we rewrite the α-fair PA problem as

$$\max_{\boldsymbol{p}} \quad \frac{1}{1-\alpha} \sum_{i=1}^{M} \left(\log_2 \left(1 + \frac{p_i}{A + \sum_{j \neq i} \beta_{j,i} p_j} \right) \right)^{1-\alpha} \tag{6}$$

$$\text{s.t.} \quad 0 \le p_i \le 1, i = 1, 2, \cdots, M, \tag{6-a}$$

where $\beta_{j,i} = \frac{G_{j,i}}{G_{i,i}}$.

On the other hand, suppose there is a connected undirected graph $G = (\mathcal{V}, \mathcal{E})$ with M vertices, where $\mathcal{V}$ is the vertex set ($|\mathcal{V}| = M$) and $\mathcal{E}$ is the edge set. We define $\beta_{i,j}$ as

$$\beta_{i,j} = \begin{cases} \infty, & \text{if } (v_i, v_j) \in \mathcal{E}, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

It is known that finding the maximum independent set of a graph is NP-hard. In the following, we show that solving the problem given in Eq.(6) is equivalent to finding the maximum independent set of G.

Consider all the $B = \binom{|\mathcal{V}|}{0} + \binom{|\mathcal{V}|}{1} + \binom{|\mathcal{V}|}{2} + \cdots + \binom{|\mathcal{V}|}{|\mathcal{V}|}$ subsets of $\mathcal{V}$, denoted by $\mathcal{B}_1, \mathcal{B}_2, \cdots, \mathcal{B}_B$, form a set $\mathcal{B}$. Here, $\binom{|\mathcal{V}|}{k}$ represents the combination operation, denoting the number of different ways of selecting k vertexes out of $|\mathcal{V}|$ vertexes. Then, according to set $\mathcal{B}$, the feasible region Ω of the problem in Eq.(6) can be partitioned into B disjoint subregions. Particularly, the kth subregion Ω_k is

$$\Omega_k = \{\boldsymbol{p} | 0 < p_i \le 1, \text{ if } v_i \in \mathcal{B}_k; p_i = 0, \text{ otherwise } \}. \tag{8}$$

Obviously, we have $\Omega_k \bigcap \Omega_m = \phi$ ($k \neq m$) and $\bigcup_{k=1}^{B} \Omega_k = \Omega$. Then, we can solve the problem in two steps:

Step 1: First, in each feasible subregion, we find a candidate point that maximizes the objective function;

Step 2: Second, among B candidate points, the one who has the largest objective function value is the optimal solution.

In the following, we discuss Step 1 and Step 2 in detail.

- Step 1
  In feasible subregion Ω_k, the candidate point is the solution to the problem given by

$$\max_{\boldsymbol{p}} \quad \frac{1}{1-\alpha} \sum_{i=1}^{M} \left(\log_2 \left(1 + \frac{p_i}{A + \sum_{j \neq i} \beta_{j,i} p_j} \right) \right)^{1-\alpha} \tag{9}$$

$$\text{s.t.} \quad 0 < p_i \le 1, \forall v_i \in \mathcal{B}_k, \tag{9-a}$$

$$p_i = 0, \forall v_i \notin \mathcal{B}_k. \tag{9-b}$$

Now, assume that there are two links v_i and v_j in $\mathcal{B}_k$ such that $(v_i, v_j) \in \mathcal{E}$. In this case, both of link i and link j get zero rate. This is because link i produces infinite interference to link j according to $\beta_{i,j} = \infty$ and vice versa. Therefore, only the isolated nodes in graph $(\mathcal{B}_k, \mathcal{E})$ have nonzero rate. Let $\mathcal{F}(\mathcal{B}_k)$ be the subset of $\mathcal{B}_k$ composed of all the isolated nodes. Now, problem in Eq.(9) becomes

$$\max_{\boldsymbol{p}} \quad \frac{1}{1-\alpha} \sum_{i \in \mathcal{F}(\mathcal{B}_k)} \left(\log_2 \left(1 + \frac{p_i}{A} \right) \right)^{1-\alpha} \tag{10}$$

$$\text{s.t.} \quad 0 < p_i \le 1, \forall v_i \in \mathcal{B}_k, \tag{10-a}$$

$$p_i = 0, \forall v_i \notin \mathcal{B}_k. \tag{10-b}$$

Obviously, the optimal value of the problem in Eq.(10) is

$$|\mathcal{F}(\mathcal{B}_k)|(1-\alpha)^{-1} \left(\log_2 \left(1 + \frac{1}{A} \right) \right)^{1-\alpha}. \tag{11}$$

- Step 2
  Among B candidate points obtained in Step 1, we need to find the one who leads to the highest objective function value. This is to solve the problem given by

$$\max_{k=1,2,\cdots,B} \quad |\mathcal{F}(\mathcal{B}_k)|(1-\alpha)^{-1} \left(\log_2 \left(1 + \tfrac{1}{A} \right) \right)^{1-\alpha}, \tag{12}$$

which is further equivalent to

$$\max_{k=1,2,\cdots,B} \quad |\mathcal{F}(\mathcal{B}_k)|. \tag{13}$$

It can be easily checked that solving the problem in Eq.(13) is equivalent to determining the size of the maximum independent set in graph $G = (\mathcal{V}, \mathcal{E})$. Hence, the proof ends. ■

As the problem in Eq.(3) is NP-hard when $0 < \alpha < 1$, we can not find the global optimality with polynomial-time complexity. In the next section, we shall focus on designing an efficient algorithm to find a local optimal point.

IV. α-FAIR POWER ALLOCATION FOR $0 < \alpha < 1$

In this section, we focus on α-Fair PA for $0 < \alpha < 1$. At the beginning, let us present two important remarks about the optimal solution of the problem in Eq.(3).

Remark 1. *In the optimal power allocation vector, there is at least one link utilizes the maximum allowed power, i.e., $p_i^{\max}$.*

Remark 1 can be explained as follows. Given any power allocation $\boldsymbol{p} = (p_1, p_2, \cdots, p_M)^T$, by increasing all the components of $\boldsymbol{p}$ by a factor η ($\eta > 1$), the rate of link i becomes

$$R_i(\eta \boldsymbol{p}) = \log_2 \left(1 + \frac{G_{i,i} p_i}{\frac{n_i}{\eta} + \sum_{j=1, j \neq i}^{M} G_{j,i} p_j} \right), \tag{14}$$

which is obviously greater than $R_i(\boldsymbol{p})$ [2]. Because α-utility is increasing in rate, we can always increase the sum-utility by increasing all components of $\boldsymbol{p}$ until one component achieves $p_i^{\max}$.

Remark 2. *Even if $R_i^{req} = 0$ for all links, the optimal rate of each link is always grater than zero for $0 < \alpha < 1$. This is because the derivative of $U_\alpha(x)$ is infinite at $x = 0$ and finite at $x > 0$.*

According to Remark 2, there always exists a positive value C_i such that the optimal rate of link i is not less than C_i. In the remainder of this paper, we replace constraint C2 of problem Eq.(3) by C2′ given by

$$\text{C2}' : R_i(\boldsymbol{p}) \ge C_i, i = 1, 2, \cdots, M, \tag{15}$$

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where the value of C_i is carefully assigned as discussed as follows.

- When $R_i^{req} > 0$, we let C_i equal to R_i^{req}. In this case, constraint C2$'$ is the same as C2 and thus the optimality does not change.
- When $R_i^{req} = 0$, according to Remark 2, we can let C_i equal to a very small positive value and the optimality also does not change [1].

It should also be pointed out that we only consider the feasible problem given in Eq.(3) [2].

In the following, we first transform the objective function of the problem in Eq.(3) into a D.C. function. Then, an algorithm is designed, followed by the discussion of the validity, optimality, convergence, parameters setting and the expansion of the proposed algorithm.

A. D.C. Formulation of the Objective Function

We note that the rate expression in Eq.(1) can be expressed as a D.C. function over power vector $\boldsymbol{p}$ [4]. This is because $R_i(\boldsymbol{p})$ can be rewritten as

$$R_i(\boldsymbol{p}) = u_i(\boldsymbol{p}) - v_i(\boldsymbol{p}), \tag{16}$$

where both $u_i(\boldsymbol{p})$ and $v_i(\boldsymbol{p})$ are concave as given by

$$u_i(\boldsymbol{p}) = \log_2\left(n_i + \sum_{j=1}^{M} G_{j,i}p_j\right) \tag{17}$$

and

$$v_i(\boldsymbol{p}) = \log_2\left(n_i + \sum_{j=1, j\neq i}^{M} G_{j,i}p_j\right), \tag{18}$$

respectively.

Then, the following theorem transforms $U_\alpha(R_i(\boldsymbol{p}))$ into a D.C. function over $\boldsymbol{p}$.

Theorem 3. *$U_\alpha(R_i(\boldsymbol{p}))$ can be written in a D.C. form. Particularly, $U_\alpha(R_i(\boldsymbol{p}))$ can be expressed as*

$$U_\alpha(R_i(\boldsymbol{p})) = U_\alpha(u_i(\boldsymbol{p}) - v_i(\boldsymbol{p})) = g_i(\boldsymbol{p}) - h_i(\boldsymbol{p}), \tag{19}$$

where $g_i(\boldsymbol{p})$ and $h_i(\boldsymbol{p})$ given respectively by Eq. (20) and Eq. (21) are concave functions.

$$g_i(\boldsymbol{p}) = U_\alpha(R_i(\boldsymbol{p})) + Z_i v_i(\boldsymbol{p}) \tag{20}$$

$$h_i(\boldsymbol{p}) = Z_i v_i(\boldsymbol{p}) \tag{21}$$

Here, in Eq.(20) and Eq.(21), Z_i is a constant that is greater than or equal to $\frac{1}{C_i^\alpha}$, i.e., $Z_i \geq \frac{1}{C_i^\alpha}$.

Proof: It can be easily derived that $U_\alpha(R_i(\boldsymbol{p})) = g_i(\boldsymbol{p}) - h_i(\boldsymbol{p})$, where $g_i(\boldsymbol{p})$ and $h_i(\boldsymbol{p})$ are given in Eq. (20) and Eq. (21), respectively. Since $v_i(\boldsymbol{p})$ is concave and Z_i is positive, $h_i(\boldsymbol{p})$ is a concave function. In the following, we show that $g_i(\boldsymbol{p})$ is also concave.

First, we note that $U_\alpha(t)$ is concave in $(C_i, +\infty)$ and the derivative of $U_\alpha(t)$ is $\frac{1}{t^\alpha}$. Then, we have

$$U_\alpha(t) \leq U_\alpha(\theta) + \frac{1}{\theta^\alpha}(t-\theta), t \geq C_i, \tag{22}$$

for all $\theta \in [C_i, \infty)$. Here, equality holds if and only if $\theta = t$.

Therefore, it can be deduced that

$$\begin{aligned} U_\alpha(t) &= \inf_{\theta\in[C_i,\infty)} \left\{U_\alpha(\theta) + \frac{1}{\theta^\alpha}(t-\theta)\right\} \\ &= \inf_{\theta\in[C_i,\infty)} \left\{U_\alpha(\theta) - \theta^{1-\alpha} + \frac{t}{\theta^\alpha}\right\}. \end{aligned} \tag{23}$$

Replacing t by $u_i(\boldsymbol{p}) - v_i(\boldsymbol{p})$, we have

$$\begin{aligned} &U_\alpha(u_i(\boldsymbol{p}) - v_i(\boldsymbol{p})) \\ =&\inf_{\theta\in[C_i,\infty)} \left\{U_\alpha(\theta) - \theta^{1-\alpha} + \frac{u_i(\boldsymbol{p}) - v_i(\boldsymbol{p})}{\theta^\alpha}\right\}. \end{aligned} \tag{24}$$

Accordingly, it can be deduced that

$$\begin{aligned} &g_i(\boldsymbol{p}) \\ =&U_\alpha(u_i(\boldsymbol{p}) - v_i(\boldsymbol{p})) + Z_i v_i(\boldsymbol{p}) \\ =&\inf_{\theta\in[C_i,\infty)} \left\{U_\alpha(\theta) - \theta^{1-\alpha} + \frac{u_i(\boldsymbol{p}) - v_i(\boldsymbol{p})}{\theta^\alpha}\right\} + Z_i v_i(\boldsymbol{p}) \\ =&\inf_{\theta\in[C_i,\infty)} \left\{U_\alpha(\theta) - \theta^{1-\alpha} + \frac{1}{\theta^\alpha}u_i(\boldsymbol{p}) + \left(Z_i - \frac{1}{\theta^\alpha}\right)v_i(\boldsymbol{p})\right\}. \end{aligned} \tag{25}$$

Then, due to $Z_i \geq \frac{1}{C_i^\alpha} \geq \frac{1}{\theta^\alpha}$ for all $\theta \in [C_i, \infty)$ and the concavity of $u_i(\boldsymbol{p})$ as well as $v_i(\boldsymbol{p})$, $g_i(\boldsymbol{p})$ can be regarded as the infimum of an infinite set of concave functions and is thus concave [15], [16]. As a result, $U_\alpha(u_i(\boldsymbol{p}) - v_i(\boldsymbol{p})) = g_i(\boldsymbol{p}) - h_i(\boldsymbol{p})$ is a D.C. function, which concludes the proof. ■

According to Theorem 3, the objective function of the α-fair PA problem can be written as

$$\begin{aligned} &\sum_{i=1}^{M} U_\alpha\left(R_i(\boldsymbol{p})\right) = \sum_{i=1}^{M}(g_i(\boldsymbol{p}) - h_i(\boldsymbol{p})) \\ =&\sum_{i=1}^{M} g_i(\boldsymbol{p}) - \sum_{i=1}^{M} h_i(\boldsymbol{p}) = g(\boldsymbol{p}) - h(\boldsymbol{p}), \end{aligned} \tag{26}$$

where both $g(\boldsymbol{p}) = \sum_{i=1}^{M} g_i(\boldsymbol{p})$ and $h(\boldsymbol{p}) = \sum_{i=1}^{M} h_i(\boldsymbol{p})$ are concave functions. Hence, the objective function of the α-fair PA problem can be presented by a D.C. function.

B. An Iterative Algorithm

Because of the D.C. objective function of the α-Fair PA problem, we can design an iterative power allocation algorithm. In particular, given the power allocation vector $\boldsymbol{p}^\zeta$ in the ζth iteration, we approximate $h(\boldsymbol{p})$ by its first-order Taylor expansion, i.e., $h\left(\boldsymbol{p}^\zeta\right) + \triangledown h^T\left(\boldsymbol{p}^\zeta\right)\left(\boldsymbol{p} - \boldsymbol{p}^\zeta\right)$, and optimize $\boldsymbol{p}$ in the $(\zeta+1)$th iteration. Specifically, $\boldsymbol{p}^{\zeta+1}$ is derived by solving problem P1 given below.

$$\text{P1}: \quad \max_{\boldsymbol{p}} \quad g(\boldsymbol{p}) - \left(h\left(\boldsymbol{p}^\zeta\right) + \triangledown h^T\left(\boldsymbol{p}^\zeta\right)\left(\boldsymbol{p} - \boldsymbol{p}^\zeta\right)\right) \tag{27}$$

$$\text{s.t.} \quad \text{C1, C2}' \tag{28}$$

[1] Actually, in practical systems, link i should be given a positive minimum required rate R_i^{req} for data transmission. Otherwise, link i can be deleted.

[2] If the required rates of links are too high, the problem given in Eq.(3) may be infeasible. We do not consider this case in this paper.

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Obviously, P1 is a convex optimization problem which can be efficiently solved by off-the-shelf solvers. The detailed algorithm is presented in Algorithm 1, in which some parameters are defined as: ζ is the iteration index, $\boldsymbol{p}^{\zeta}$ is the power vector after the ζth iteration and ε is the maximum tolerance for the termination of the algorithm.

Algorithm 1 An iterative algorithm

1: **Initialization**
- Set $\boldsymbol{p}^0$, ε;
- Set $\zeta = 0$;

2: **repeat**
3: Solve Problem P1 and derive $\boldsymbol{p}^*$;
4: $\zeta = \zeta + 1$;
5: $\boldsymbol{p}^{\zeta} = \boldsymbol{p}^*$;
6: **until** $|\boldsymbol{p}^{\zeta} - \boldsymbol{p}^{\zeta-1}| \leq \varepsilon$

C. Discussion of the Proposed Algorithm

We now discuss the validity, convergence, optimality of the proposed algorithm as well as the choice of Z_i.

1) Validity: In each iteration in the proposed algorithm, we approximate the second concave function of the objective, i.e., $h(\boldsymbol{p})$, by its first-order Taylor expansion. Note that $h(\boldsymbol{p})$ is actually the sum of M log functions of an affine combination of $\boldsymbol{p}$. Therefore, this approximation is very close to $h(\boldsymbol{p})$ in a relatively large neighbourhood of a given $\boldsymbol{p}^{\zeta}$. The good performance of this kind of approximation is also explained in [4] and [13].

2) Convergence and Optimality: Similar to [13] and [17], we can easily prove that the proposed algorithm always converges to a local optimal point of the primal α-Fair PA problem. Comparing with high computation complexity to get the global optimality, we obtain a local optimal (or global optimal in a two-link case) point with low complexity as shown in Section V [3].

3) The Choice of Z_i: The value of Z_i needs to be carefully determined in the proposed algorithm.

First, as illustrated in the proof of Theorem 3, Z_i should be large enough to guarantee the concavity of $g_i(\boldsymbol{p})$ given in Eq.(20).

Second, too large Z_i is also not a good choice for the following reasons. In the $(\zeta + 1)$th iteration, the objective function is

$$\begin{aligned} & g(\boldsymbol{p}) - \left(h\left(\boldsymbol{p}^{\zeta}\right) + \bigtriangledown h^T\left(\boldsymbol{p}^{\zeta}\right)\left(\boldsymbol{p} - \boldsymbol{p}^{\zeta}\right)\right) \\ =& \sum_{i=1}^{M}\left\{U_\alpha(R_i(\boldsymbol{p})) + Z_i\left(v_i(\boldsymbol{p}) - v_i(\boldsymbol{p}^{\zeta}) - \bigtriangledown v_i^T(\boldsymbol{p}^{\zeta})(\boldsymbol{p} - \boldsymbol{p}^{\zeta})\right)\right\}. \end{aligned} \tag{29}$$

Note that $Z_i\left(v_i(\boldsymbol{p}) - v_i(\boldsymbol{p}^{\zeta}) - \bigtriangledown v_i^T(\boldsymbol{p}^{\zeta})(\boldsymbol{p} - \boldsymbol{p}^{\zeta})\right)$ is the dominant part of the objective function when Z_i is sufficiently large. Because $v_i(\boldsymbol{p})$ is concave over $\boldsymbol{p}$, we have $v_i(\boldsymbol{p}) - v_i(\boldsymbol{p}^{\zeta}) - \bigtriangledown v_i^T(\boldsymbol{p}^{\zeta})(\boldsymbol{p} - \boldsymbol{p}^{\zeta}) \leq 0$, where equality holds if and only if $\boldsymbol{p} = \boldsymbol{p}^{\zeta}$. Therefore, maximizing Eq.(29) will result in the $\boldsymbol{p}^{\zeta+1}$ that is very close to $\boldsymbol{p}^{\zeta}$. In other words, the algorithm will converge very slowly.

As a result, in practice, we set $Z_i = \frac{1}{(C_i)^{\alpha}}$.

4) Expansion: The core method of the proposed algorithm can also be applied to a general concave and non-decreasing utility function $U(x)$ only by properly setting Z_i. In the following, we discuss the choice of Z_i in two cases.

- $U'(0) = \infty$
  In this case, Remark 2 is still valid and there always exists a positive C_i such that the optimal rate of link i is not less than it. Then, similar to the proof of Theorem 3, we can set $Z_i = U'(C_i)$ and the sum-utility given by
  $$\sum_{i=1}^{M} U(R_i(\boldsymbol{p})) = \sum_{i=1}^{M} U(u_i(\boldsymbol{p}) - v_i(\boldsymbol{p})) \tag{30}$$
  can be expressed in a D.C. form, where the first and the second concave functions are given respectively by
  $$g(\boldsymbol{p}) = \sum_{i=1}^{M} U(u_i(\boldsymbol{p}) - v_i(\boldsymbol{p})) + \sum_{i=1}^{M} Z_i v_i(\boldsymbol{p}) \tag{31}$$
  and
  $$h(\boldsymbol{p}) = \sum_{i=1}^{M} Z_i v_i(\boldsymbol{p}). \tag{32}$$
- $U'(0) < \infty$
  In this case, by setting $C_i = \max\{R_i^{req}, 0\}$ and $Z_i = U'(C_i)$, we can make sure that the expression in Eq.(30) is a D.C. function, where the first and the second concave functions are given by Eq.(31) and Eq.(32), respectively.

As a result, the proposed algorithm in Algorithm 1 can be utilized to carry out power allocation for the corresponding sum-utility maximization.

V. Simulation Results

This section first shows that the proposed algorithm can achieve the global optimal point in a two-link case. Then, some results are presented in 10-link cases. Note that in the following simulation, the noise power is $n_i = 0.1\mu$W and the maximum allowed transmit power is $p_i^{\max} = 1m$W for all links. The initial point of the proposed algorithm is set as $\boldsymbol{p}^0 = \boldsymbol{p}^{\max}$.

A. Optimality in a Two-Link Case

We consider two links in the network and the channel gain is

$$\mathbf{G} = \begin{bmatrix} 0.4310 & 0.0605 \\ 0.0002 & 0.3018 \end{bmatrix}. \tag{33}$$

The minimum required rates of the two links are $R_1^{req} = R_2^{req} = 0.5$bps.

According to Remark 1, in the optimal power allocation, either link 1 or link 2 transmits with the maximum allowed power. Therefore, we observe the optimal power allocation by two one-dimensional searchings as shown in Fig.1. First, by letting $p_1 = p_1^{\max}$, we plot the sum-utility by varying p_2

[3]The proposed algorithm may also achieve the global optimality, although we can not verify it [13].

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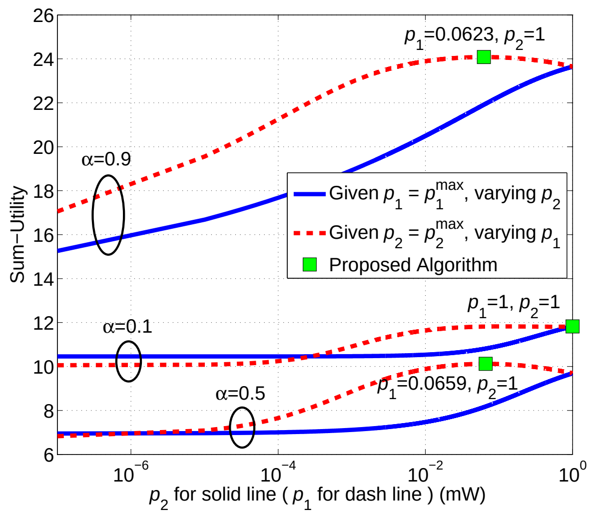

Fig. 1. Optimality illustration with 2 links existing in the network.

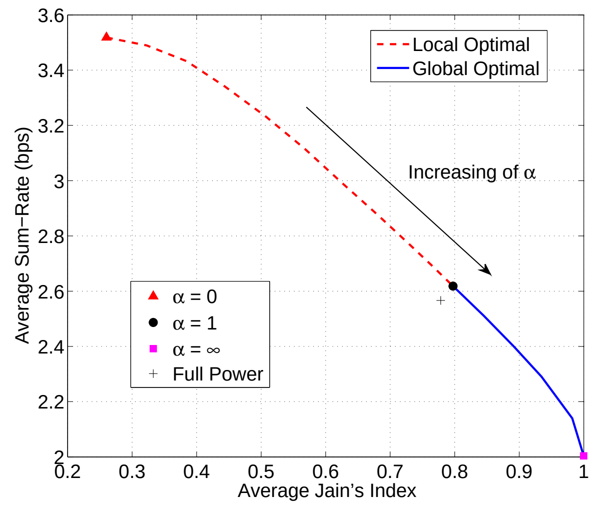

Fig. 2. The tradeoff between sum-rate and Jain's index by α-fair power allocation.

(solid line in Fig.1). Second, by setting $p_2 = p_2^{\max}$, we plot the sum-utility by varying p_1 (dash line in Fig.1). Then, the highest point of the two curves indicates the optimal power allocation and the optimal sum-utility.

We also simulate our proposed algorithm by letting $Z_i = 0.5^{-\alpha}$ for all links, where the achieved power allocation is marked by green squares in Fig.1. Based on this figure, we can see that the proposed algorithm achieves the global optimal power allocation.

B. Performance in 10-Link Cases

Consider there are $M = 10$ links in the network. We randomly pick 20 network realizations and simulate the α-fair PA algorithm by varying α from zero to infinity, where ε is set as $\varepsilon = 0.5 \times 10^{-3}$. Assume each link has the same minimum required rate $R_i^{req} = 0.1$bps. For a given α and a given network setting, we utilize Jain's index to evaluate the degree of fairness. Here, the Jain's index is defined as

$$\text{Jain's Index} = \frac{\left(\sum_{i=1}^{M} R_i\right)^2}{M \sum_{i=1}^{M} R_i^2}, \tag{34}$$

which is bounded in $[\frac{1}{M}, 1]$ [18]. Because its value can be interpreted as the fraction of favored links, a larger Jain's index is more fair [9], [19].

Figure 2 presents the tradeoff curve between the averaged sum-rate and the averaged Jain's index across the 20 network instances. The results in Fig.2 are derived in many ways according to the value of α:

- $\alpha = 0$: by the algorithm in [4];
- $\alpha = 1$: by the convex optimization modeled in [3];
- $\alpha = \infty$: by the linear programming modeled in [3];
- $1 < \alpha < \infty$: by the convex optimization modeled in Eq.(4) in this paper;
- $0 < \alpha < 1$: by the proposed algorithm in Algorithm 1 in this paper.

These results in Fig.2 can be illustrated as follows. 1) We string the three special cases $\alpha = 0$, $\alpha = 1$ and $\alpha = \infty$ and obtain a smooth tradeoff between sum-rate and Jain's index. In particular, if we increase α, a more fair power allocation can be obtained; otherwise, a larger sum-rate can be got. 2) In the interval $1 < \alpha < \infty$, the achieved curve is already global optimal. 3) In the interval $0 < \alpha < 1$, it is credible that good performance is achieved because it has a smooth transition from the global optimal interval $\alpha \geq 1$ to the case $\alpha = 0$.

In Fig.3, we also present the averaged iteration times of the proposed algorithm for $0 \leq \alpha < 1$. It can be seen that the proposed algorithm converges to a local optimal point by hundreds of iterations. This complexity is comparable with (more exactly, 2-4 times) the complexity of the algorithm for the case $\alpha = 0$ in [4]. This a little increase of complexity comes from the fact that, the case $0 < \alpha < 1$ corresponds to maximizing the sum of concave utility of rate while the case $\alpha = 0$ represents the sum-rate maximization. Obviously, the problem for the case $0 < \alpha < 1$ is more complex than the case $\alpha = 0$, although both of them are NP-hard.

VI. Conclusion

In multi-link one-tone spectrum-sharing networks, we proved that the α-fair PA problem is convex when $1 < \alpha < \infty$, which indicates that a standard convex solver can find the global optimality with polynomial complexity. We also showed that the problem is NP-hard when $0 < \alpha < 1$. To deal with this difficult case, based on the D.C. formulation of the objective function, we designed an iterative algorithm which can efficiently converge to a local optimal point. Simulation results showed that the global optimality is achieved in two-link cases. In addition, a smooth tradeoff between sum-rate and Jain's index is obtained by varying α from zero to infinity.

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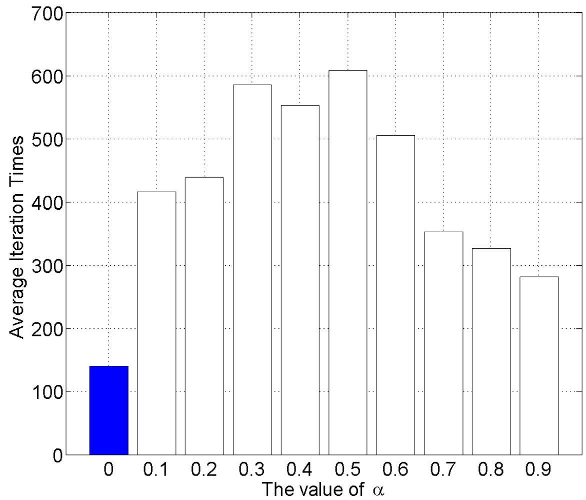

Fig. 3. The average iteration times of the proposed algorithm.

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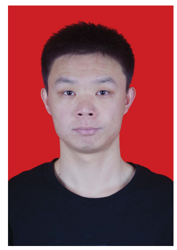

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